# The Renormalization Group & la Gellmann-Low

(27) $\mathcal{L}_R + \mathcal{L}_{C.T.} = \mathcal{L}_{BARE}$ ← "physical" as it encodes the UV boundary condition (that's why it depends on $\Lambda$ or $\frac{1}{\epsilon}$).

$\mathcal{L}_R$: finite but unphysical; $\mathcal{L}_{C.T.}$: singular; $\mathcal{L}_{BARE}$: singular but physical

in the sense of arbitrary separation of finite terms

[and notice that the physical results should not depend on how we shuffle the finite terms around]

and, by construction, $\mathcal{L}_{Bare}$ in Eq.(27) delivers finite correlator.

[Aside: one can (not must!!) choose a scheme where $\mathcal{L}_R$ is constructed in terms of "physical" couplings, e.g. $M_{2\to2}$ as measured at some kinematical config. would allow us in $\phi^4$ theory to have $\mathcal{L}_R = -\frac{\lambda_R \phi^4}{4!}$ with $M_{2\to2}(E=\bar{E}) = -\lambda_R$. Or one can renormalize the mass to the physical one, $m_R^2 = m^2_{phys.}$. While this is certainly ok, it's just an arbitrary choice out of infinitely many that are equally valid, (such as e.g. picking $\lambda_R$ at some unphysical kinematic of $M_{2\to2}$). Moreover, it would just remain as arbitrary as keeping around the "$\mu$" in dim reg, since one would have $\bar{E}$ as new parameter, and certainly physics does not depend on whether we decide to define the coupling as $M_{2\to2}(E=\bar{E})$ rather than to some other $E \neq \bar{E}$.
In other words, $\mu \Leftrightarrow \bar{E}$, no much of a difference then!
And in fact it's often very useful, as we will see, not to stick to "physical" choice of the renormalized couplings or masses, especially because it leads to premature break down of perturbation theory due to "large logarithm" (see Weinberg ch. 18 for an account) e.g. $M_{2\to2}(E=\bar{E}) = \lambda_R \Rightarrow M_{2\to2}(E) = -\lambda_R\left(1 + \frac{\lambda_R}{16\pi^2}\log\left(\frac{s t u}{(\bar{E})^6}\right)\right)$, the second term becoming larger than tree-level $-\lambda_R$ for large $\log(E/\bar{E})$ even if $\frac{\lambda_R}{16\pi^2} \ll 1$. [This can be solved below via RG-equation applied to $\mu = \bar{E}$ and recalling that $\lambda_R = \lambda(\bar{E}) = \lambda(\mu)$ and varying $\bar{E} = \mu$]]

- **Exemple:** $\mathcal{L}_R = (\partial\phi)^2 + m^2\phi^2 + \frac{\lambda}{4!}\phi^4\mu^{4-D=\epsilon}$ in dim-reg.

(28) $$\mathcal{L}_{C.T} = \frac{\delta Z}{2}(\partial\phi)^2 + \frac{\delta m^2}{2}m^2 + \frac{\delta\lambda\,\mu^\epsilon}{4!}\phi^4$$

(29) $$\mathcal{L}_R + \mathcal{L}_{CT} = \underbrace{(1+\delta Z)}_{Z}(\partial\phi)^2 + \frac{(m^2+\delta m^2)}{2}\phi^2 + \frac{(\lambda+\delta\lambda)\mu^\epsilon}{4!}\phi^4$$

(30) $$\Rightarrow \mathcal{L}_{BARE} = (\partial\phi_0)^2 + m_0^2\frac{\phi_0^2}{2} + \frac{\lambda_0}{4!}\phi_0^4 \quad \text{with}$$

(31) $$\begin{cases} Z = 1+\delta Z \\ m_0^2 = (m^2+\delta m^2)/Z \\ \lambda_0 = (\lambda+\delta\lambda)\mu^\epsilon/Z^2 \\ \sqrt{Z}\,\phi = \phi_0 \leftarrow \text{bare fields} \end{cases} \quad \text{bare mass \& couplings w/.} \quad \begin{cases} \delta Z = \left(\frac{\lambda}{16\pi^2}\right)\frac{\#}{\epsilon} + \left(\frac{\lambda}{16\pi^2}\right)^2\frac{\#}{\epsilon^2} + \dots \\ \delta\lambda = \left(\frac{\lambda}{16\pi^2}\right)\frac{\#}{\epsilon} + \dots \\ \delta m^2 = \dots \end{cases}$$

We can perform equivalent splittings of the same, given, $\mathcal{L}_{BARE}$

as physical results do not depend on how we shuffle the finite terms (at fixed $\mathcal{L}_{Bare}$, which "defines" the theory).

In dim-reg the freedom of the choice of finite terms is represented by the choice of $\mu$ (see e.g. $M_{2\to2} = -\lambda\left(1+\frac{\lambda}{32\pi^2}\log\frac{stu}{\mu^6}\right)$ where changing $\mu\to\mu e^\alpha$ is like redefining $\lambda_R=\lambda$ by the finite shift $\lambda_R \to \lambda_R + \frac{3}{16\pi^2}\lambda_R^2\alpha$ )

Aside: in a different scheme, e.g. where we defined $\lambda$ via $M_{2\to2}(\epsilon=\bar\epsilon)=\lambda_R$, this freedom of changing the finite terms is associated to changing the $\bar\epsilon$ where one defines the $\lambda_R$. In the hard cutoff case one subtract $\log\Lambda^2/\mu^2$ and the undetermined finite constant $\mu$ is back. It is necessarily so because C.T. are defined only up to finite terms.

(32) $$\boxed{\mathcal{L}_R + \mathcal{L}_{C.T.} = \mathcal{L}_R' + \mathcal{L}_{C.T}' \quad (= \mathcal{L}_{BARE})}$$

(like in Wilsonian RG identif. $\Lambda_{wilson}=\mu$)

in dim-Reg $\mathcal{L}_R = \mathcal{L}_R(\lambda(\mu))$ ; $\mathcal{L}_{C.T}(\lambda(\mu))$ ($\leftarrow$ to compensate $\kappa \gtrsim \mu$ )

— Example: we calculated in $\phi^4$ in $D=4$ at 1-loop in dim-reg —

(33) $$M_{2\to 2} = -\lambda \left[ 1 + \frac{\lambda}{32\pi^2} \left( 3\gamma_E - 6 + \ln \frac{stu}{(4\pi\mu^2)^3} - i\pi \right] + \dots \right.$$

$$\simeq -\lambda - \frac{6\lambda^2}{32\pi^2} \log \frac{E}{\mu}$$

( $\leftarrow$ Those constants too can be absorbed in $\mu \to \bar{\mu}$ )

Since physics should be independent of $\mu$ $\Rightarrow$ $M_{2\to 2}$ can't depend separately on $\lambda$ & $\mu$. (In fact, working to this 1-loop order it depends only on the combination $\hat{\lambda} \equiv \lambda - \frac{6\lambda^2}{32\pi^2}\log\mu$ , $M_{2\to 2} = -\hat{\lambda} - \frac{6\hat{\lambda}^2}{16\pi^2}\log E$)

So we see that $(\lambda, \mu)$ is a redundant pair !!

$$\boxed{\lambda = \lambda(\mu) \text{ such that } \frac{d}{d\ln\mu} Obs = 0}$$

(again like Wilsonian RG. $\Lambda_{wilson} \to \mu$)

for example:

(34) $$\boxed{\frac{d}{d\ln\mu} M_{2\to 2} = 0 \Rightarrow +\frac{d\lambda}{d\ln\mu} = \frac{6\lambda^2}{32\pi^2} = \frac{3}{16\pi^2}\lambda^2 = \beta_{\phi^4_{D=4}}}$$

In another scheme, e.g. $\lambda = \lambda(\bar{E})$ defined by $M_{2\to 2}(E=\bar{E}) = -\lambda\left(1 + \frac{\lambda}{32\pi^2} \log \frac{stu}{\bar{E}^6}\right)$, we see that $(\lambda, \bar{E})$ is not an indep. pair, manifestly, and $\frac{d}{d\bar{E}} obs = 0$ such as Eq.(34) $\to$ $\frac{d}{d\ln\bar{E}} M_{2\to 2} = 0 \Rightarrow -\beta + \frac{3\lambda^2}{16\pi^2} = 0$ OK!!

We could have obtained the same demanding that the bare parameters, i.e. those that define $\mathcal{L}_{BARE}$ (that define the theory at hand) are $\mu$-indep.

## Example: $\phi^4$ at one-loop in $D=4-\epsilon$

(35) $$\lambda_0 = \mu^\epsilon \frac{(\lambda+\delta\lambda)}{(1+\delta Z)^2} \underset{\substack{\text{1-loop}\\ D=4-\epsilon}}{=} \mu^\epsilon \lambda \Big(1 + \underbrace{3\frac{\lambda}{16\pi^2}\frac{1}{\epsilon}}_{\delta\lambda \text{ we calculated previously}}\Big) \Rightarrow$$

(36) $$\mu^{-\epsilon}\frac{d\lambda_0}{d\ln\mu} = \epsilon\lambda\left(1+\frac{3\lambda}{16\pi^2}\frac{1}{\epsilon}\right) + \beta\left(1+\frac{3\lambda}{16\pi^2}\frac{1}{\epsilon}\right) + 3\frac{\lambda\beta}{16\pi^2}\frac{1}{\epsilon} = 0$$

(37) $$\Rightarrow \quad \beta\left(1+\frac{6\lambda}{16\pi^2}\frac{1}{\epsilon}\right) + \epsilon\lambda + \frac{3\lambda^2}{16\pi^2} = 0 \Rightarrow \text{to lowest order } \beta = -\epsilon\lambda$$

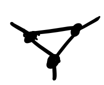
plug back in

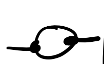
(38) $$\beta - \frac{6\lambda^2}{16\pi^2} + \epsilon\lambda + \frac{3\lambda^2}{16\pi^2} = 0 \Rightarrow \boxed{\beta = -\epsilon\lambda + \frac{3\lambda^2}{16\pi^2} \qquad \text{ok}}$$

## Example: $\frac{g}{3!}\phi^3$ in $D=6-\epsilon$ dim.

(39) $$[\phi^3] = 3\left(\frac{D-2}{2}\right) \Rightarrow [g] = D - [\phi^3] = \frac{6-D}{2} \Rightarrow g_0 = \mu^{\epsilon/2}\cdot\left[\frac{(g+\delta g)}{(1+\delta Z)^{3/2}}\right]$$

(40) where we calculated (see Eq. 20 L6/P6)

$$\delta g = g\cdot\frac{g^2}{(4\pi)^3}\left(-\frac{1}{\epsilon}\right) \text{ from }$$ 

$$\delta Z = \frac{g^2}{(4\pi)^3}\left(-\frac{1}{\epsilon}\right)\frac{1}{6} \text{ from }$$ 

$$\Rightarrow g_0 = \mu^{\epsilon/2} g \Big(1 - \frac{g^2}{(4\pi)^3}\frac{1}{\epsilon}\overbrace{\left(1-\tfrac{1}{4}\right)}^{3/4}\Big) + \mathcal{O}(g^4)$$

(41) $$\mu^{-\epsilon/2}\frac{d}{d\ln\mu} g_0 = 0 = \frac{\epsilon}{2} g\left(1 - \frac{3}{4}\frac{g^2}{(4\pi)^3}\frac{1}{\epsilon}\right) + \beta\Big(\overbrace{1 - 3\frac{g^2}{(4\pi)^3}\frac{3}{4}\frac{1}{\epsilon}}^{+\frac{9}{8}\frac{g^3}{(4\pi)^3}}\Big)$$

to lowest order $\beta = -\frac{\epsilon g}{2}$

plug back-in

(42) $$\beta = -\frac{\epsilon}{2} g + \frac{3}{8}\frac{g^3}{(4\pi)^3} - \frac{9}{8}\frac{g^3}{(4\pi)^3} = \boxed{-\frac{\epsilon}{2} g - \frac{3}{4}\left(\frac{g^3}{(4\pi)^3}\right) = \beta_{\phi^3}}_{D=6-\epsilon}$$

Notice that $\beta_{\phi^3} < 0$ in $D=6$ (the theory is asymptotically free in the UV)
we explain this later

## Multiple couplings in various dimensions to all orders

With multiple couplings, even with non-renormalizable (i.e. irrelevant) or relevant interactions, we can repeat the strategy by definining dimensionless renormalized couplings in terms of which the dimensionful bare couplings are expressed

(43) $$\begin{cases} g_{n,bare} = \mu^{(D-\Delta_{\theta_n})} \cdot \left( g_n(\mu) + \dfrac{a_1^{(n)}(g)}{\epsilon} + \dfrac{a_2^{(n)}(g)}{\epsilon^2} + \dots \right) \\ \mathcal{L}_{int}^{Bare} = \sum_n g_{n,bare}\, \theta_{n\,bare}, \quad [\theta_{n\,bare}] = \Delta_{\theta_n}(D) \\ (\epsilon = \bar{D} - D) \end{cases}$$

dimensionful ($g_{n,bare}$); dimensionless ($g_n(\mu)$)

$\Delta_{\theta_n}(D)$: function of $D$, e.g. $[\phi^4] = 2D-4$

$\boxed{a_i^{(n)}: \text{pole } i\text{-th order in } g_{n,bare}}$

where $\epsilon = \bar{D} - D$, $\bar{D}$ being the integer dim via aim at ($\bar{D}=4$ in $\phi^4$, $\bar{D}=6$ in $\phi^3$)

It's thus convenient to pull out the $\mu^{\bar{D}-\Delta_{\theta_n}(\bar{D})}$-factor

(44) $$g_{n,bare} = \mu^{\overbrace{\bar{D}-\Delta_{\theta_n}(\bar{D})}^{\text{classical dim of } g_n \text{ at } D=\bar{D}}} \cdot \mu^{\alpha_n \epsilon} \left( g_n(\mu) + \frac{a_1^{(n)}(g)}{\epsilon} + \frac{a_2^{(n)}(g)}{\epsilon^2} + \dots \right)$$

Since $\Delta_{\theta_n}(D)$ is linear in $D$.

$(\alpha_n \cdot \epsilon + \bar{D} - \Delta_{\theta_n}(\bar{D}) = \alpha_n(\bar{D}-D) + \bar{D} - \Delta_{\theta_n}(\bar{D}) = D - \Delta_{\theta_n}(D))$

(example: $\alpha = +1$ in $\phi^4_{D=4}$; $\alpha = +1/2$ in $\phi^6_{D=6}$)

For marginal couplings $\bar{D} - \Delta_{\theta_n}(\bar{D}) = 0$ whereas it is positive and negative for relevant and irrelevant couplings respectively.

By demanding $\mu$-independence of bare couplings, $\boxed{\mu \frac{d}{d\mu} g_{bare} = 0}$

we extract the $\beta$-functions:

(45) $$0 = \mu^{-(\bar{D} - \Delta_{\theta_n}(\bar{D}) + \alpha_n \epsilon)} \frac{d\, g_{n,bare}}{d\ln\mu} =$$

$$a_i^{(n,m)} \equiv \frac{\partial a_i^{(n)}}{\partial g_m} \qquad \frac{d\, g_m}{d\ln\mu}$$

Kronecker

$$= (\bar{D} - \Delta_{\theta_n}(\bar{D}) + \alpha_n \epsilon)\left[g_n + \frac{a_1^{(n)}}{\epsilon} + \dots\right] + \underbrace{\left(\delta_n^m + \frac{a_1^{(n,m)}}{\epsilon} + \dots\right)}_{\text{inverting this matrix}} \beta_m$$

(46)
$$\begin{cases} \beta_n = -\left[\delta_n^m - \frac{a_1^{(n,m)}}{\epsilon} + \dots\right]\left[g_m + \frac{a_1^{(m)}}{\epsilon} + \dots\right](\bar{D} - \Delta_{\theta_m}(\bar{D}) + \alpha_m \epsilon) \\[2ex] = \underbrace{-\alpha_n g_n \epsilon}_{\mathcal{O}(\epsilon^1)} \underbrace{- (\bar{D} - \Delta_{\theta_n}(\bar{D})) g_n + \alpha_m g_m a_1^{(n,m)} - \alpha_n a_1^{(n)}}_{\mathcal{O}(\epsilon^0)} + \underbrace{\frac{1}{\epsilon}(\dots) + \frac{1}{\epsilon^2}(\dots) + \dots}_{\mathcal{O}(1/\epsilon^{n\geq 1})} \end{cases}$$

Now, a finite change in $\mu$ require a change in $g_n(\mu)$ to compensate it: must be finite

$$\Rightarrow \boxed{\frac{1}{\epsilon}(\dots) + \frac{1}{\epsilon^2}(\dots) + \dots = 0}$$ all must vanish identically $\Rightarrow \beta$ is finite as $\epsilon \to 0$

(47) $$\boxed{\beta_n = -(\bar{D} - \Delta_{\theta_n}(\bar{D})) g_n + \alpha_m g_m \frac{\partial a_1^{(n)}}{\partial g_m} - \alpha_n a_1^{(n)}}$$ (sum over m)

All order result!!

classical: just from tree-level $g_{n,bare}\big|_{tree} = \mu^{\bar{D} - \Delta_{\theta_n}(\bar{D})} g_n$

quantum!

Remark:
* This is a very universal formula: the $\beta$-function in a generic theory in a generic dimension $D = \bar{D}$
* it depends only on the simple poles $\frac{a_1^{(n)}}{\epsilon}$ in $g_{n,bare}$

higher poles $\frac{a_{m\geq 2}^{(n)}}{\epsilon^{m\geq 2}}$ do not matter!!

check-1

$\frac{\lambda\phi^4}{4!}$ in $D$-dimensions has $\alpha = 1$ and $D - \Delta_{\phi^4} = 4 - D$

(48) $$\beta_{\phi^4} = -(4-D)\lambda \underbrace{+\lambda\frac{\partial}{\partial\lambda}a_1 - a_1}_{\lambda^2\frac{\partial}{\partial\lambda}(a_1/\lambda)} = -(4-D)\lambda + \lambda^2\frac{\partial}{\partial\lambda}(a_1/\lambda)$$

$\Downarrow$

(49) $$\begin{cases} \beta_{\phi^4} = -(4-D)\lambda + \lambda^2\frac{\partial}{\partial\lambda}(a_1/\lambda) \\ \frac{a_1}{\lambda} = 3\frac{\lambda}{16\pi^2} + \#_2\left(\frac{\lambda}{16\pi^2}\right)^2 + \#_3\left(\frac{\lambda}{16\pi^2}\right)^3 + \dots \end{cases}$$

← all order result for $\phi^4$-theory

2-loop correct. to $\frac{1}{\epsilon}$ c.T. (the $\frac{1}{\epsilon^2}$ irrelevant)

3-loop corr. to $1/\epsilon$ c.T. ($\frac{1}{\epsilon^3}$ irrelevant)

$\boxed{\frac{1}{\epsilon^2}, \frac{1}{\epsilon^3}, \dots \text{ c.T. do not matter}}$

(50) $$\boxed{\beta^{\text{1-loop}}_{\phi^4} = -(4-D)\lambda + \frac{3\lambda^2}{16\pi^2}}$$

in agreement w./ Eq.(38)

check-2

$g\frac{\phi^3}{3!}\mu^{\epsilon/2}$ in $D$-dim.

$\alpha = 1/2$

$D - \Delta_{\phi^3} = D - \frac{3(D-2)}{2} = \frac{6-D}{2} = \epsilon$

(51) $$\Rightarrow \begin{cases} \beta_{\phi^3} = -\frac{\epsilon}{2}g + \frac{1}{2}g^2\frac{\partial}{\partial g}(a_1/g) \\ \frac{a_1}{g} = -\frac{3}{4}\frac{g^2}{(4\pi)^3} + \#_2\left(\frac{g^2}{(4\pi)^3}\right)^2 + \#_3\left(\frac{g^2}{(4\pi)^3}\right)^3 + \dots \end{cases}$$

← all order result in $\phi^3$-th.

2-,3-loop corr. to $\frac{1}{\epsilon}$-term in c.T.

(again, $\frac{1}{\epsilon^2}, \frac{1}{\epsilon^3}, \dots$ not important)

$\Downarrow$

(52) $$\boxed{\beta_{\phi^3}\Big|_{\text{1-loop}} = -\frac{\epsilon g}{2} - \frac{3}{4}\cdot\frac{g^3}{(4\pi)^3}}$$

**Remark:** (a) The fact that $1/\epsilon, \frac{1}{\epsilon^2}, \dots$ terms in Eq. (46) vanish, i.e. finiteness of $\beta$ when $\epsilon \to 0$, it's actually powerful because it fixes higher order poles $\frac{a^{(n)}_{i=2,\dots}}{\epsilon^i}$ from $a^{(n)}_1$ only.

(b) Take for example a single coupling like in $\phi^4$-theory for simplicity and keep track of $O(\frac{1}{\epsilon^2})$-terms in RG-equation

(53)
$$\begin{cases} 0 = \mu^{-\epsilon}\frac{d}{d\ln\mu}\lambda_{bare} = \epsilon\cdot\lambda\left(1+\sum\frac{a_i/\lambda}{\epsilon^i}\right)+\beta\left(1+\sum\frac{\cancel{a_i/\lambda}}{\epsilon^i}+\sum\frac{(a_i'-\cancel{a_i/\lambda})}{\epsilon^i}\right) \\ \beta = -\frac{\epsilon\cdot\lambda\left(1+\sum\frac{a_i/\lambda}{\epsilon^i}\right)}{\left(1+\sum a_i'/\epsilon^i\right)} = -\epsilon\lambda+\lambda^2\frac{\partial}{\partial\lambda}(a_1/\lambda)+\frac{1}{\epsilon}\left(\lambda\frac{\partial}{\partial\lambda}^2(a_2/\lambda)-\lambda\frac{\partial}{\partial\lambda}^2(a_1/\lambda)\cdot\frac{\partial a_1}{\partial\lambda}\right)+\frac{1}{\epsilon^2}(\dots) \end{cases}$$

exp in loop counting (not $\epsilon$)

$\Rightarrow$ requiring $\frac{1}{\epsilon}(\dots)=0$ set a differential equation for $a_2/\lambda$

(54)
$$\boxed{\frac{\partial}{\partial\lambda}(a_2/\lambda) = \underbrace{\frac{\partial}{\partial\lambda}(a_1/\lambda)\,\frac{\partial a_1}{\partial\lambda}}_{\text{Known}}} \Rightarrow \text{solving for } a_2/\lambda$$

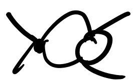
Likewise one determines $a_{n\geq 2}/\lambda$ too from setting to zero $\frac{1}{\epsilon^{n\geq 2}}(\dots)$

From $a_1$-alone one can extract all $a_n$'s, it's just a matter of knowing $a_1/\lambda = \left(\frac{3\lambda}{16\pi^2}\right)+\#_2\left(\frac{\lambda}{16\pi^2}\right)^2+\dots$ to higher loop

For example, in $\frac{\phi^4}{4!}$-theory in $D=4$, $a_1 = \frac{3\lambda^2}{16\pi^2}$ so that

(55)
$$a_2/\lambda = A\left(\frac{\lambda}{16\pi^2}\right)^2 \overset{\text{Eq. (54)}}{\Longrightarrow} 2A\cdot\frac{\cancel{\lambda}}{(16\pi^2)^2} = \frac{3}{(16\pi^2)}\cdot\frac{3}{(16\pi^2)}\cdot 2\cancel{\lambda} \Rightarrow \boxed{A=9}$$

i.e. we determined $a_2 = 9\lambda^3/(16\pi^2)^2$ from $a_1$ at 1-loop.

This corresponds diagrammatically to  being the square of  giving rise to $1/\epsilon^2$ (or $\log^2\Lambda$ in Pauli-Villars or hard cutoff) whereas  and  are genuinely new, $O\left(\frac{\lambda^3}{(16\pi^2)^2}\right)\frac{1}{\epsilon}$, that contribute to $a_1$ at 2-loop.

The RG guarantees this in generality.

We can also see it at the level of the amplitude:

(c) From Weinberg theorem, the $\frac{1}{\epsilon^n}$ pole correspond to the leading powers of $\log^n E$ $\Rightarrow$ the leading Log's are determined!

Example:

(56) $$M_{2\to2}\Big|_{\phi^4\text{-th.}} = -\lambda\Big(1 + \underbrace{\overbrace{\frac{3\lambda}{16\pi^2}\log E/\mu}^{\text{leading log}}}_{\text{1-loop}} + \left(\frac{\lambda}{16\pi^2}\right)^2 \underbrace{\Big[\overbrace{A\log^2 E/\mu}^{\text{leading log}^2} + \overbrace{B\log E/\mu}^{\text{sublead.}} + C\Big]}_{\text{2-loop}}$$

from new/GT. at 2-loop

(57) $$\Rightarrow \frac{d}{d\ln\mu}M_{2\to2} = 0 = -\beta\Big(\cancel{1} + 2\cdot\frac{3\lambda}{16\pi^2}\log(E/\mu) + \underbrace{3\left(\frac{\lambda}{16\pi^2}\right)^2\Big[A\log^2 E/\mu + \dots\Big]}_{\text{can neglect}}\Big)$$

$$+\cancel{3\frac{\lambda^2}{16\pi^2}} - \frac{\lambda^3}{(16\pi^2)^2}\Big[-2A\log E/\mu - B\Big]$$

(58) since $\beta = \frac{3\lambda^2}{16\pi^2} + b^{(2)}\frac{\lambda^3}{(16\pi^2)^2} + \dots \Rightarrow$

$\boxed{A = 9 \qquad b^{(2)} = B}$

leading $\log^2$ already determined at 1-loop $\beta$-func.

subleading log determined by 2-loop $\beta$-funct.

So we see that the running coupling allows to systematically determine the higher order Log's. This is a general feature of RG-equations and it's quite important because it allows to keep perturbation theory (P.T.) under control even with exp. separated energy scales

# RG-flow and its Asymptotics

Now, that we see that higher powers of large-$\log(E/\mu)$ can be determined via RG, we can address the asymptotic RG-evolution both @

$E \gg \mu \; (\gg m)$ or $(m \ll) \; E \ll \mu$ $\qquad t=-s-t$

In particular, let's consider some observable Obs:

($\mu$ can be the $\mu = E$ in the "physical" choice of ren. coupl.

$$(59) \qquad Obs = \underset{\text{tree}}{g}^{\#}\left(1 + \frac{g^2}{16\pi^2}\underbrace{\ln(E/\mu)}_{\gg 1 \, ?} + \left(\frac{g^2}{16\pi^2}\right)^2 \underbrace{\log(E/\mu)^2}_{\gg 1 \, ?} + \dots\right)$$

$\frac{g^2}{16\pi^2} \ll 1$ would not seem enough for P.T. when $\log E/\mu \gtrsim \frac{16\pi^2}{g^2}$ ?

Can we recover control when $\frac{g^2}{16\pi^2} \ll 1$ but $\frac{g^2}{16\pi^2}\log E \gg 1$ ? Yes!

The idea is that for any given $E$ we can change the splitting $\mathcal{L}_e + \mathcal{L}_{C.T}$ so that $(\lambda(\mu), \mu)$ is chosen with $\mu \simeq E \implies$ no large logarithm appears

Then, by moving by infinitesimal steps $\mu \to \mu + d\mu$ one can change to another $\mu \simeq E$ from the initial one.

In formulas:

(60) $$\boxed{\frac{dg}{d\ln\mu} = \beta(g) \Longrightarrow \ln E/\mu = \int_{g(\mu)}^{g(E)} \frac{dg}{\beta(g)}}$$ ← no large logs in it

$\beta = g\left(\#_1 \frac{g^2}{16\pi^2} + \#_2\left(\frac{g^2}{16\pi^2}\right)^2 + \dots\right)$

just powers of $g^2/16\pi^2$

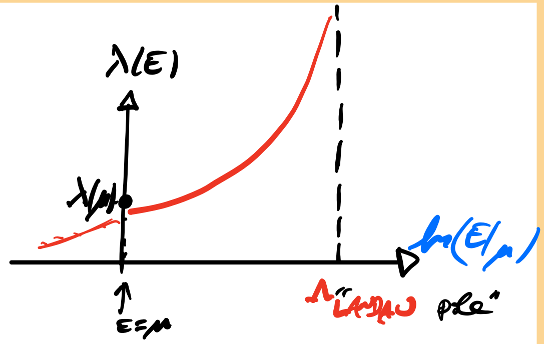
Example: $\frac{\lambda}{4!}\phi^4$ in $D=4$ $\longrightarrow$ $\beta_{\phi^4}\big|_{D=4,\ 1\text{-loop}} = \frac{3\lambda^2}{16\pi^2}$

(61) $$\ln E/\mu = \int_\mu^E \frac{d\lambda}{3\lambda^2/16\pi^2} = \left(\frac{16\pi^2}{3}\right)\cdot\left(\frac{1}{\lambda(\mu)} - \frac{1}{\lambda(E)}\right)$$

(62) $$\lambda(E) = \frac{\lambda(\mu)}{1 - \frac{3}{16\pi^2}\lambda(\mu)\log(E/\mu)}$$

As long as $\lambda(E)$ remains small we can write $M_{2\to2}(E \gg \mu) = -\lambda(E) + O\left(\frac{\lambda(E)^2}{16\pi^2}\right)$

(63) $$M_{2\to2}(E \gg \mu) \simeq -\lambda(E) = \frac{-\lambda}{1 - \frac{3\lambda}{16\pi^2}\log(E/\mu)}$$




and, if the log's are small that we can expand it back (just to see what the RG-equation (59) is resumming)

(64) $$M_{2\to2}(E) = -\lambda\cdot\left[1 + \left(\frac{3\lambda}{16\pi^2}\right)\log(E/\mu) + \left(\frac{3\lambda}{16\pi^2}\right)^2 \log^2(E/\mu) + \dots\right]$$

it nicely matches the $A=9$ in Eq.(55−58), but with no effort.

The $-\lambda\left(\frac{\lambda}{16\pi^2}\right)^n \text{Log}^n(E/\mu)$ coefficients are predicted to be $3^n$. Subleading $\lambda^{n+1}\log^n$ can be predicted once $\beta$ is calculated to next loop order, and so on.

On the other hand, given a finite coupling $\lambda$ at some scale $\mu$, the coupling become large at a finite energy scale $\Lambda_{LANDAU}$

(65) $$\boxed{\Lambda_{LANDAU} \sim \exp\left[\frac{16\pi^2}{3\lambda(\mu)}\right]\cdot\mu}$$

$\mu$ being the scale of the initial condition $\lambda=\lambda(\mu)$.

The theory is no longer perturbative at finite energy: it can't be extrapolated past $E\sim\Lambda_{LANDAU}$. $\Rightarrow$ $\boxed{\phi^4 \text{ in } D=4 \text{ is EFT}}$

So this means that $\phi^4$-theory should be regarded as an EFT with cutoff not larger than $\Lambda_{LANDAU}$. In this sense, $\phi^4$-theory is not renormalizable, and one should expect higher operators, e.g. $\frac{\phi^6}{\Lambda^2_{LANDAU}}+\dots$ We still usually call it "perturbatively" renormalizable because the Landau-pole scale is exponentially large than $\mu$.

Conversely, for a fixed $\lambda(\mu_0)$ at $\mu_0 \gg E$, the coupling $\lambda(E)$ goes to zero as $E/\mu_0 \to 0$

(66) $$\boxed{\lambda(E) \xrightarrow[E/\mu_0\to 0]{} 0}$$

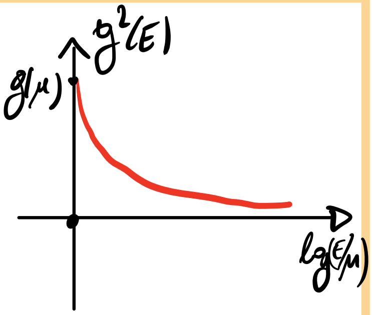
"Triviality" of $\phi^4$-theory (in $D=4$) (in the "continuum limit" à la Wilson)

This is called triviality of $\phi^4$ theory if one really tries to send $\mu_0 \to \infty$ keeping $\lambda(\mu_0)$ fixed, as that corresponds from a Wilsonian perspective to try to remove the cutoff regulator while keeping the bare coup. fixed. $\Rightarrow$ $\phi^4$ can't be defined beyond perturbation theory.

Example: $g\frac{\phi^3}{3!}$ in $D=6$ $\longrightarrow \beta_{\substack{\phi^3\\ D=6}} \underset{\text{1-loop Eq.(52)}}{=} -\frac{3}{4}\left(\frac{g^3}{(4\pi)^3}\right)$

(65) $$\ln(E/\mu) = \int_\mu^E \frac{dg}{\beta(g)} = \frac{(4\pi)^3}{3/4}\cdot\frac{1}{2}\left[\frac{1}{g^2(E)} - \frac{1}{g^2(\mu)}\right]$$

(66) $$\boxed{g^2(E) = \frac{g^2(\mu)}{1+\frac{3}{2}\left(\frac{g^2(\mu)}{(4\pi)^3}\right)\log(E/\mu)}}$$




$\longrightarrow 0$ as $E/\mu \to \infty$ $\Longleftarrow$ Asymptotic freedom in the UV

Because $g^2(E\to\infty)\to 0$ there is no obstruction in extrapolating this theory to arbitrary short-distance (i.e. removing the cutoff, or taking the cont. limit)
QCD & Yang-Mills are also asymptotically free theories!

[disclaimer: $\phi^3$ in $D=6$, contrary to QCD & Y.M. in $D=4$, does not in fact makes sense fully non-perturbatively because of vacuum instability... but for $g$ small enough to tunneling rate to a new vacuum is exp. small... We are using $\phi^3$ just as toy model]].

Conversely, the coupling become large in the IR: $g^2(E\to\Lambda_{IR}) \gg (4\pi)^3$ $\longrightarrow$ form bound state?

(67) $$\boxed{\Lambda_{IR} = \Lambda_{UV}\cdot\exp\left(-\frac{2}{3}\frac{(4\pi)^3}{g(\Lambda_{UV})}\right)}$$

expon. separated by $\Lambda$!!

A new IR mass scale linked to pair $(g(\Lambda),\Lambda)$ given in the UV: Dim. transmutation

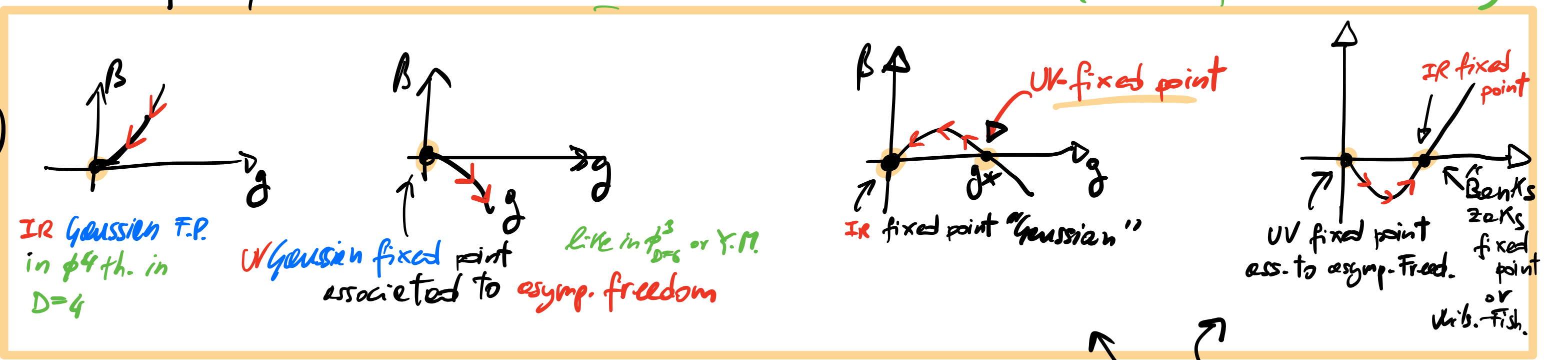
# Fixed Points

← as example of asymptotics

We have seen that $\beta > 0$ leads to coupling growing (decreasing) in the UV (IR) and viceverse for $\beta < 0$

(arrows point towards IR)

(68)




There could be however a situation where $\beta \to 0$ for some $g \to g_*$

If the running mass vanish at $g = g_*$, it's called a Fixed point

(69) $$\boxed{\beta(g_*) = 0}$$

$(m = 0)$

when this happens $g(\mu) = g_* = \text{const}$

no run anymore.

(With several couplings $\beta_i(g_1, g_2, \dots) = 0$ all simultaneously.)

If $\beta$ has simple zero around $g_*$, ← linear in $(g - g_*)$ so $\beta'_*$ can be considered, comparing w/. (47) $\beta = -(D-\Delta)g$

(70) $$\underbrace{\beta}_{\text{around F.P. } g = g_*} = \beta'_* (g - g_*) + \dots$$

(↖ or with more couplings $\beta_i = \left.\frac{\partial \beta_i}{\partial g_j}\right|_{g = g_*} (g - g_*)_j + \dots$)

linear in $g - g_*$, so that comparing to Eq. (47) $\beta = -(D - \Delta)g + \dots$ we can, (redefining $g - g_* = \tilde{g}$ $\beta_{\tilde{g}} = \beta'_* \tilde{g}$), call $\beta'_*$ the quantum or anomalous dimension of $g$. (With multiple couplings, in general, need to diagon. $\left.\frac{\partial \beta_i}{\partial g_j}\right|_{g = g_*}$)

This is further justified by integrating the RG-flow near $g_*$:

(71) $$\ln E/\mu = \int_{g(\mu)}^{g(E)} \frac{dg}{\beta(g)} \Big|_{g\sim g_*} \simeq \int_{g(\mu)}^{g(E)} \frac{dg}{\beta_*'(g-g_*)} = \frac{1}{\beta_*'} \ln\left|\frac{g(E)-g_*}{g(\mu)-g_*}\right|$$

that is

(72) $$(g(E)-g_*) = \text{const.}\ \underbrace{(E/\mu)^{\beta_*'}}_{\text{simple power-law scaling}} \cdot \begin{cases} \xrightarrow{E/\mu\to\infty} 0 & \beta_*'<0 \quad \text{UV-fixed P.} \ (3^{rd}\text{ Fig in (68)}) \\ \xrightarrow{E/\mu\to 0} 0 & \beta_*'>0 \quad \text{IR-fixed P.} \ (4^{th}\text{ fig. in (68)}) \end{cases}$$

simple power-law scaling controlled by $\beta_*'$.

This should be contrasted with the free theory $\beta = -\underbrace{(D-\Delta)}_{\text{classical dim. of coup.}} g + \dots$ where the $g_*=0$ and it's reached by the classical dimension set by $D-\Delta$.

If $g_*/\text{loop} \ll 1$ we can approach the F.P. within perturbation theory.

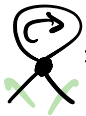
---

Example: Wilson-Fisher Fixed point in $D=4-\epsilon$ dimensions

The toy-model theory we consider now is $\phi^4$-theory with a mass term in $D=4-\epsilon$ dimensions with $\epsilon \neq 0$! (in fact in stat. mech. this is used to go to $\epsilon \to 1$ & approach the 3D Ising model...)

We have two "couplings" $(\lambda, m^2)$ already calculated $\beta_\lambda = -\epsilon\lambda + \frac{3\lambda^2}{16\pi^2}$ ← need $\beta_{m^2}$

(73) $$m_0^2 \equiv \mu^2 g_{2,\text{bare}} = \mu^2 \frac{(g_2+\delta g_2)}{(1+\delta Z)} \underset{\text{1-loop}}{=} \mu^2(g_2+\delta g_2) \to \text{need the mass C.T. } \delta g_2$$

(74) $$\boxed{\Sigma_{\text{self en.}} =} $$  $$\boxed{= -\frac{\lambda}{4!}\mu^{4-D}\, 4\cdot 3 \cdot \int \frac{d^D k}{(2\pi)^D}\frac{1}{(k^2+m^2)} = \frac{\lambda}{16\pi^2}\, m^2 \frac{1}{\epsilon} + \dots}$$

(e.g. by $\frac{\partial}{\partial m^2}$ first, see the log-div, extract $1/\epsilon \leftrightarrow \log\Lambda$, int. back)

$$\left(\widetilde{W}^{(2)} = (p^2+m^2-\Sigma)\right) \Rightarrow \boxed{m^2_{\substack{\text{1-loop}\\ \text{MS}}} = \frac{1}{\epsilon}\frac{\lambda}{16\pi^2}m^2 = \left(\frac{1}{\epsilon}\frac{\lambda}{16\pi^2}g_2\right)\mu^2}$$

(75) so $$\begin{cases} a_1^{(g_2)} = \dfrac{\lambda}{16\pi^2}\, g_2 \\[2ex] a_1^{(\lambda)} = \dfrac{3\lambda}{16\pi^2}\cdot\lambda \end{cases} \qquad g_n = \begin{pmatrix} g_2 \\ g_4=\lambda \end{pmatrix},\quad g_{n,\text{bare}} = \begin{pmatrix} \mu^2\left(g_2 + \frac{a_1^{(g_2)}}{\epsilon} + \dots\right) \\ \mu^\epsilon\left(\lambda + \frac{a_1^{(\lambda)}}{\epsilon} + \dots\right) \end{pmatrix}$$

and using Eq.(47) (w/ $\alpha_{g_2}=0$, $\alpha_\lambda = 1$), the mass has always the can. dim since $[m]=[\partial]$)

(Eq.47 again) $$\boxed{\beta_n = -\left(\bar{D} - \Delta\theta_n(\bar{D})\right)g_n + \alpha_m\, g_m \frac{\partial a_1^{(n)}}{\partial g_m} - \alpha_n\, a_1^{(n)}}$$

(76) $$\boxed{\beta_i = \begin{pmatrix} \beta_\lambda \\ \beta_{g_2} \end{pmatrix} = \begin{pmatrix} -\epsilon\lambda + \dfrac{3\lambda^2}{16\pi^2} \\[2ex] -2g_2 + \dfrac{\lambda g_2}{16\pi^2} \end{pmatrix} = \begin{pmatrix} -\epsilon + \dfrac{3\lambda}{16\pi^2} & 0 \\[2ex] \dfrac{g_2}{16\pi^2} & -2 \end{pmatrix}\begin{pmatrix} \lambda \\ g_2 \end{pmatrix}}$$

(77) $$\begin{cases} \beta_\lambda = 0 \\ \beta_{g_2} = 0 \end{cases} \Rightarrow \begin{array}{l} \lambda_* = \dfrac{16\pi^2}{3}\epsilon \quad (\ll 1 \text{ if } \epsilon \ll 3/16\pi^2) \\[2ex] \dfrac{g_2^*\lambda_*}{16\pi^2} - 2g_2^* = g_2^*(\epsilon-2) = 0 \end{array} \to \boxed{\begin{pmatrix} \lambda_* = \dfrac{16\pi^2}{3}\epsilon \\[1ex] g_2^* = 0 \end{pmatrix} = g_i^*}$$

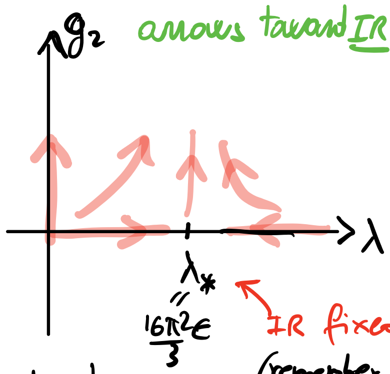
Wilson-Fisher fixed point

(78) $$\left.\frac{\partial\beta_i}{\partial g_j}\right|_{g_i=g_i^*} = \begin{pmatrix} -\epsilon + \dfrac{6\lambda_*}{16\pi^2} & 0 \\[2ex] \dfrac{g_2^*}{16\pi^2} & -2 + \dfrac{\lambda_*}{16\pi^2} \end{pmatrix} = \begin{pmatrix} \overbrace{\epsilon}^{>0} & 0 \\ 0 & \underbrace{-2+\epsilon/3}_{<0} \end{pmatrix}$$

changed sign wrt Gaussian

$\epsilon/3$ is the mass anom. dim

(79) $\Rightarrow$ near WF-F.P. $\begin{cases} \beta_\lambda \simeq \epsilon(\lambda-\lambda_*) \\ \beta_{g_2} = (-2+\epsilon/3)g_2 \end{cases}$




since mass remain relevant, need to tune it to zero to reach WF-F.P.

notice that $\phi^4$ is marginally relevant at the Gaussian F.P. in $D=4-\epsilon$ ($\beta_\lambda = -\epsilon\lambda + \dots$ $\epsilon>0$) but it becomes irrelevant at the Wilson-Fisher F.P. (the mass instead remains relevant at both)

The scaling of $g_2$ in the Gaussian F.P. would be $\beta_{g_2} = -2g_2$
(that is $g_2(\mu) = g_2(\mu_0)(\mu_0/\mu)^2$, which is a complicated way of saying
$m^2_{bare} = g_2(\mu)\mu^2 = g_2(\mu_0)\mu_0^2$)
At the Wilson-Fisher F.P. instead, using (79), we get

coeff. in Gaussian F.P.

(80) $$\boxed{g_2(\mu) = g_2(\mu_0)\left(\frac{\mu_0}{\mu}\right)^{2(1-\epsilon/6)}}$$

the mass is still relevant but less so for $\epsilon>0$

mass anom. dim. $\hookrightarrow$ in Wilson-Fisher F.P.

More generally, the mass anomalous dim. is defined by taking out the classical piece

(81) $$-\frac{1}{2}\frac{1}{m^2}\frac{dm^2}{d\ln\mu} = -\frac{1}{2}\frac{1}{m^2}\frac{d\mu^2 g_2}{d\ln\mu} = -\frac{1}{2}\left(2+\mu^2\underbrace{\beta_2}_{=-2g_2+(\dots)}\right) = -\frac{1}{2}(\dots) \underset{\text{F.F.}}{=} -\frac{\epsilon}{6}$$

Once one extracted the asymptotic behavior of coupling & masses, the asymptotic behavior of other observables is easy to extract.
Consider for example, an observable with a certain mass dimension

(82) $\quad Obs = \mathcal{O}(E, g_i(\mu), m(\mu), \mu) = E^{\Delta} \cdot F(\underbrace{\tfrac{E}{\mu}}_{\text{could give large log's}}, g_i(\mu), \tfrac{m(\mu)}{E})$ $\quad \Delta = [\mathcal{O}]$

minimize the effect of large logs by choosing $\mu \simeq E$

(83) $$\boxed{Obs = E^{\Delta} F(1, g_i(E), \cancel{\tfrac{m(E)}{E}}) \xrightarrow[E \text{ large}]{} E^{\Delta}\Big[\underbrace{g_i^{\#}(E)}_{\text{set leading } E\text{-scaling}} + \underbrace{\# \tfrac{g_i^2(E)}{16\pi^2}}_{\text{if } \ll 1} + \dots\Big]}$$

no large log. $\quad$ tree level order

(e.g. cross-section in $\phi^4$-th. $\sigma(E \gg E_0) \propto \frac{1}{E^2}\lambda^2(E) = \frac{1}{E^2}\frac{\lambda^2(E_0)}{\left(1 - 3\frac{\lambda^2(E_0)}{16\pi^2}\ln\frac{E}{E_0}\right)^2}$
not just naive $1/E^2$ scaling.

Caveat: neglecting mass can be an issue when there are IR-div. ⇒ need to look at IR-safe observables

Incidentally, from the RG equations of couplings & masses one can extract RG-eq. for observables

(84) $$\mu \frac{d}{d\mu} Obs = 0 = \left(\mu \frac{\partial}{\partial \mu} + \beta_i \frac{\partial}{\partial g_i}\right) Obs = 0$$

← if including masses among couplings, otherwise add $-\gamma_m m^2 \frac{\partial}{\partial m^2}$

## Callan - Symanzik Equation

In addition to $\frac{d\, Obs}{d \ln\mu} = 0$ or $\frac{d}{d \ln\mu} g_i^{bare} = 0$, $\frac{d}{d \ln\mu} m^2_{bare} = 0$ one can consider how correlation functions scale with $\mu$.

(84) $$\boxed{\langle \phi_{0_1} \cdots \phi_{0_n} \rangle = \prod_{i=1}^{n} \sqrt{Z_i}\, \underbrace{\langle \phi_1 \cdots \phi_n \rangle}_{\text{renorm. correlators}}}$$

↑ bare fields

$\phi_{0_i} = \sqrt{Z}\, \phi_i$

so that $\mu \frac{d}{d\mu} \langle \text{bare correlator} \rangle = 0$ gives

(85) $$\boxed{\left( \sum_i \gamma_i + \mu \frac{d}{d\mu} \right) \langle \phi_1 \dots \phi_n \rangle = 0}$$

total der.

$$\boxed{\gamma_i \equiv \frac{1}{2} \frac{d \ln Z_i}{d \ln \mu}}$$

Example: in $\phi^3$ in $D=6$, Eq. (40)(42)

$\gamma_{1\text{-loop}} = \frac{1}{2} \cdot \frac{1}{6} \left( \frac{-1}{\epsilon} \right) \cdot \frac{g}{(4\pi)^3} \cdot 2 \left( -\epsilon \frac{g}{2} + \dots \right)$

$= \frac{1}{2} \cdot \frac{g^2}{(4\pi)^3} \cdot \frac{1}{6}$ ok

or also

Callans-Symanzik equation

(86) $$\boxed{\left( \sum_i \gamma_i + \mu \frac{\partial}{\partial \mu} + \beta_i \frac{\partial}{\partial g_i} \right) \langle \phi_1 \dots \phi_n \rangle}$$

partial der.

again including masses among coupl., if not add $\gamma_{m^2} \frac{\partial}{\partial \ln m^2}$

The formal solution of (86) or (85) is obtained by just recalling (84)

(87) $$G^{(n)}(\mu, g(\mu), p) \cdot \Pi \sqrt{Z_i} = \mu\text{-indep.}$$

defining $\left( \langle \phi_1 \dots \phi_n \rangle \equiv G^{(n)}(\mu, g(\mu), p) \right)$

(88) $$\boxed{G^{(n)}(\mu, g(\mu), p) = G^{(n)}(\mu', g(\mu'), p) \cdot \frac{\Pi_i \sqrt{Z_i(g(\mu'))}}{\Pi_j \sqrt{Z_j(g(\mu))}}}$$

So we need first to solve for the $Z$ dependence on $\mu$. Let's work for simplicity with a single field and a single coupling:

(89) $$\gamma = \frac{1}{2} \frac{d \ln Z}{d \ln \mu} \Rightarrow \ln \sqrt{\frac{Z(\mu)}{Z(\mu_0)}} = \int_{\mu_0}^{\mu} \gamma(g(\mu)) \, d \ln \mu = \int_{g_0}^{g(\mu)} \frac{\gamma(g)}{\beta} dg$$

change variables $\frac{dg}{d \ln \mu} = \beta$

(90) $$\sqrt{Z(\mu)} = \sqrt{Z(\mu_0)} \exp \left[ \int_{g_0}^{g} \frac{\gamma(g)}{\beta(g)} dg \right]$$

(91) $$\boxed{G^{(n)}(\mu, g(\mu), p) = \exp\left[-n\int_{g_0}^{g} \frac{\gamma(g)}{\beta(g)}\, dg\right] \cdot G^{(n)}(\mu_0, g(\mu_0), p)}$$

Notice that at a fixed point $g = g^* = \text{const}$ so that from (89)

(92) $$\gamma\big|_{\text{F.P. at } g=g_*} = \gamma_* \Rightarrow \boxed{\sqrt{\frac{Z(\mu)}{Z(\mu_0)}} = \left(\mu/\mu_0\right)^{\gamma_*}}$$ ← simple power-law scaling at F.P.

(93) $$\boxed{G^{(n)}(\mu, g(\mu), p)\Big|_{g=g^* \text{ F.P.}} = \left(\mu/\mu_0\right)^{-n\gamma_*} \cdot G(\mu_0, g^*, p)}$$

Scaling correlation at a Fixed Point

**Example:** 2pt − $\langle \phi_1 \phi_2 \rangle$ near a fixed point

From eq. (86) $\Rightarrow$ $$\left(\mu\frac{\partial}{\partial\mu} + 2\gamma_*\right) G^{(2)}(x_1, x_2, \mu) = 0$$

$$G^{(2)}(x_1, x_2, \mu) = \frac{1}{|x_1-x_2|^2}\, \bar{G}^{(2)}(|x_1-x_2|\mu)$$ ← translations + rotations + dim. analysis (D=4)

(otherwise pull out $\frac{1}{|x_1-x_2|^{D-2}}\bar{G}^{(2)}$) in D=4 →

(94) $$\left(\mu\frac{\partial}{\partial\mu} + 2\gamma_*\right)\bar{G}^{(2)}(|x_1-x_2|\mu) = 0 \Rightarrow \boxed{x\cdot\frac{\bar{G}^{(2)\prime}(x)}{\bar{G}^{(2)}(x)} = -2\gamma_*}$$

$\Downarrow$

$$G(x) = \text{const}\; x^{-2\gamma_*}$$

(95) $$G^{(2)}(x_1, x_2, \mu)\Big|_{\text{F.P.}} = \frac{1}{|x_1-x_2|^2}\cdot\frac{\text{const.}}{(|x_1-x_2|\mu)^{2\gamma_*}}$$

$$= \text{const}/|x_1-x_2|^{2(\underset{\text{classic dim.}}{1}+\gamma_*)}$$

it scales like if $[\phi] = 1+\gamma_*$ justifying the name of $\gamma$.

One can reach the same conclusion in mom. space from the solution in Eq. (93)

(96) $\hat{G}^{(2)} = \hat{G}^{(2)}(\mu, g_*, p) = \left(\mu/\mu_0\right)^{-2\gamma_*} \hat{G}^{(2)}(\mu_0, g_*, p)$

in mom. space

Eq. (93)

$$= \left(\mu/\mu_0\right)^{-2\gamma_*} \cancel{\frac{1}{p^2}} F(p^2\mu_0^2) = \cancel{\frac{1}{p^2}} F(p^2\mu^2)$$

(97) $\Rightarrow$ F homogeneous fnc. degree $+\gamma_*$ $\quad F(x) = x^{+\gamma_*}$

(98) $\hat{G}^{(2)} = \frac{1}{p^2} \cdot \frac{1}{(p^2\mu^2)^{-\gamma_*}} \xrightarrow{\text{F.T.}} \boxed{G^{(2)} \propto \frac{1}{|x_1 - x_2|^{2(1+\gamma_*)}}}$

Finally, a slightly more pedestrian derivation based on perturb. th. would be to minimize the large logs.

(99) $\hat{G}^{(2)}(\mu, g_*, p) = \left(\mu/\mu_0\right)^{-2\gamma_*} \hat{G}(\mu_0, g_*, p)$

$\hookrightarrow$ choose $\mu_0 \simeq p$ so to minimize large logs

$$= \left(\frac{\mu}{\mu_0}\right)^{-2\gamma_*} \frac{1}{p^2}\left[1 + O\left(\frac{g_*^2}{16\pi^2} \ln p/\mu_0\right) + \dots\right]$$

$$\propto \frac{1}{p^{2(1-\gamma_*)}} \xrightarrow{\text{F.T.}} \frac{1}{|x_1 - x_2|^{2+2\gamma_*}}$$